About the Recursive Decomposition of the lattice of co-Moore Families

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Abstract. Given a set $U_n = \{0, 1, ..., n-1\}$, a collection $\mathcal{M}$ of subsets of U_n that is closed under intersection and contains U_n is known as a Moore family. The set of Moore families for a fixed n is in bijection with the set of Moore co-families (union-closed families containing the empty set) denoted itself $\mathbb{M}_n$. In this paper, we propose for the first time a recursive definition of the set of Moore co-families on U_n. These results follow the work carried out in [1] to enumerate Moore families on U_7.

1 Introduction

In this article we study the set of Moore co-families of a given universe $U_n = \{0, 1, ..., n-1\}$. The concept of a Moore family, or of a closure operator (extensive, isotone and idempotent function on 2^{U_n}), or of a complete implicational system, is applied in numerous fields. For example, let us consider mathematical researches such as [2] for algebra and computer science researches such as [3] for order theory and lattices, [4] for relational databases and finally [5] and [6] for data analysis and formal concept analysis. The name 'Moore family' was first used by Birkhoff in [7] referring to E.H. Moore's early century researches in [8]. Technically, a Moore family on U_n, denoted by $\mathcal{M}$, is a collection of sets (or a family) closed under intersection and containing U_n.

The set of Moore families on U_n is itself a Moore family or a closure system (a closure system being the set of the fixed points of a closure operator). Thus, the system composed of Moore families contains one maximum element (2^{U_n}, all subsets of U_n) and the intersection of two

Moore families is a Moore family itself. To get an overall view of the properties of this closure system, see [9].

The problem of enumerating Moore families on n elements is a complex issue for which there is no known formula ([10]). Even the absence of such a formula has not been proved. Numerous combinatorial problems fall in the same case: for example, the number of monotone Boolean functions known as the Dedekind number. In [11], Burosch considers the issue of counting Moore families as natural, so he suggests an upper bound for that number (see also [12]). An often supported approach to understand such a formula involves counting the number of objects for the first values of n using a systematic procedure. Thus, the number of Moore families was computed for $n = 7$ ([1]). We can find such integer sequences on the well-known On-line Encyclopedia of Integer Sequences [1].

In [1] authors count Moore co-families on U_7. For that, they highlight some structural properties of the Moore co-families lattice. So, this new article presents a first theoretical study of these properties. Particularly, we give a decomposition theorem of the Moore co-families lattice based on an operator $h()$ and we study several properties of this operator.

So, the remainder of the report is divided into two main sections. The first section is devoted to the recursive decomposition theorem, and the second section deals with the properties of the $h()$ operator. An assessment of this work and its approach is provided in the conclusion. Finally, the set of demonstrations of intermediate properties is appended.

In the rest of the paper, we denote elements by numbers $(1, 2, 3, \dots)$. Sets are denoted by capital letters $(A, B, C, \dots)$. Families of sets are denoted by calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots)$. Finally, we denote the sets of families of sets by black board letters $(\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots)$.

2 Recursive decomposition of the Moore co-families lattice

In the introduction, we have defined a Moore family on U_n as a collection of sets containing U_n and closed by intersection. For reasons of legibility of the results and simplification of expressions, the remainder of the report will deal with the study of the set of families closed by union and containing the empty set called the Moore co-families and denoted by $\mathbb{M}_n$. Moore families are in fact in bijection with this set. For a given Moore family,

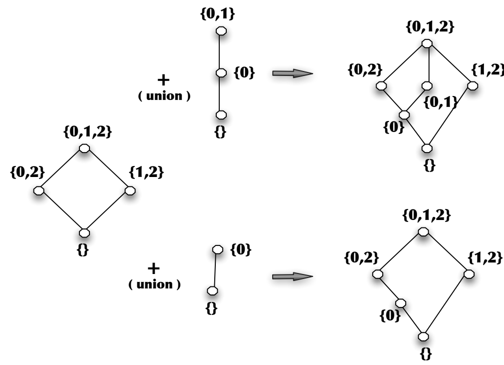
[1] http://www.research.att.com/ njas/sequences/A102896

one only has to complement every set to obtain a Moore co-family. For example, the Moore family $\{\{0\},\{0,1\},\{0,2\},\{0,1,2\}\}$ on U_3 corresponds to the Moore co-family $\{\emptyset,\{1\},\{2\},\{1,2\}\}$ and vice versa.

2.1 Study of the relationships between $\mathbb{M}_n$ and $\mathbb{M}_{n-1}$

A Moore co-family $\mathcal{M}$ on U_n can be decomposed into two parts. The part consisting of the sets of $\mathcal{M}$ containing the element $(n-1)$ (denoted by $\mathcal{M}_{up}$ for the upper part), and the complementary part (denoted by $\mathcal{M}_{low}$ for the lower part). The empty set is added to $\mathcal{M}_{up}$ in order to be present in both parts. Naturally, $\mathcal{M}=\mathcal{M}_{up}\cup\mathcal{M}_{low}$. In the one hand the family $\mathcal{M}_{low}$ is clearly a family of $\mathbb{M}_{n-1}$. On the other hand, the family $\mathcal{M}_{up}$ is a Moore co-family on U_n with the peculiarity that all its nonempty sets contain the element $(n-1)$ (we will denote by $\mathbb{M}_n^{up}$ the set of Moore co-families on U_n having this property). Thus, $\mathcal{M}_{up}$ can be seen as a Moore co-family of $\mathbb{M}_{n-1}$ for which the element $(n-1)$ has been added to each set.
Example: let $\mathcal{M}$ be the family on U_3, $\{\emptyset,\{0\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}$, we can decompose this family into two sub Moore co-families: $\mathcal{M}_{low}=\{\emptyset,\{0\},\{0,1\}\}$ and $\mathcal{M}_{up}=\{\emptyset,\{0,2\},\{1,2\},\{0,1,2\}\}$.

To study the matching conditions between a family in $\mathbb{M}_{n-1}$ and a family in $\mathbb{M}_n^{up}$, we define the notion of a **compatible** family. Thus, we will say that a Moore co-family in $\mathbb{M}_{n-1}$ is compatible with a Moore co-family in $\mathbb{M}_n^{up}$ if and only if the union of the two families is a Moore co-family in $\mathbb{M}_n$. The example in the following figure illustrates that for a fixed upper part, there are several **compatible** lower parts.



On the left, a family in $\mathbb{M}_3^{up}$ (all sets contain the element 2) to which are associated two different compatible Moore co-families in $\mathbb{M}_2$. In the two cases the family obtained, on the right, is a Moore co-family in $\mathbb{M}_3$.

In [1] we have shown that for a given upper family $\mathcal{M}_{up}$, there exists a unique **maximal compatible family**. The latter is moreover such that all Moore co-families compatible with $\mathcal{M}_{up}$ are exactly its sub Moore co-families. More formally:

Proposition 1. *Let* $\mathcal{M}_{up}$ *in* $\mathbb{M}_n^{up}$*. Then there exists a unique Moore co-family* $\mathcal{M}_{max}$ *on* U_{n-1} *such that:* $\forall \mathcal{M} \in \mathbb{M}_{n-1}$ *the two following assertions are equivalent :*

1. $\mathcal{M}$ *is compatible with* $\mathcal{M}_{up}$*;*
2. $\mathcal{M} \subseteq \mathcal{M}_{max}$*.*

For example, the maximal Moore co-family compatible with $\mathcal{M}_{up}$ (cf. figure above) is the Moore co-family $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. It can be verified that the two compatible families given are in fact sub-families of this family.

The function $f : \mathbb{M}_n^{up} \rightarrow \mathbb{M}_{n-1}$ defined below allows to characterize the maximal compatible family of a Moore co-family belonging to $\mathbb{M}_n^{up}$.

Definition 1. *One defines the function* $f : \mathbb{M}_n^{up} \rightarrow \mathbb{M}_{n-1}$ *such that* $f(\mathcal{M}) = \{X \in 2^{U_{n-1}} \mid \forall M \in \mathcal{M} \setminus \emptyset,\ M \cup X \in \mathcal{M}\}$.

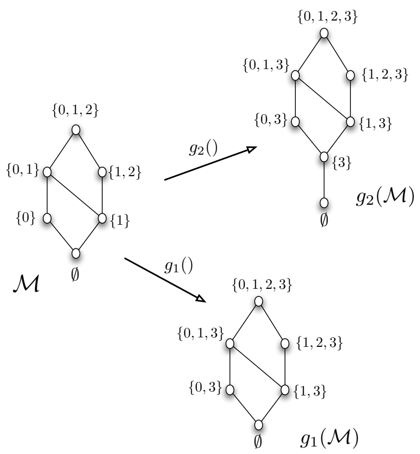
To summarize, let us remember that the set of families in $\mathbb{M}_{n-1}$ compatible with a given family $\mathcal{M}$ in $\mathbb{M}_n^{up}$ has a maximal family called $\mathcal{M}_{max}$ equal to $f(\mathcal{M})$ and that there exists a 'greedy' algorithm to compute it ([1]). Moreover all Moore co-families contained in $\mathcal{M}_{max}$ are compatible with $\mathcal{M}$. By noting $\downarrow \mathcal{X}$ the principal ideal based on $\mathcal{X}$ in $\mathbb{M}_{n-1}$, the set of families compatible with $\mathcal{M}$ coincides with $\downarrow \mathcal{M}_{max}$, in other words, with $\downarrow f(\mathcal{M})$.

2.2 Recursive decomposition theorem

In fact, the set $\mathbb{M}_n^{up}$ can be partitioned into two sets: the set $\mathbb{M}_n^{up_1}$ of families that do not contain the singleton $\{n-1\}$ and the set $\mathbb{M}_n^{up_2}$ of families that contain it. These two sets are in natural bijection with $\mathbb{M}_{n-1}$. More formally, functions g_1 and g_2 associate with each family $\mathcal{M}$ in $\mathbb{M}_{n-1}$ a family in $\mathbb{M}_n^{up_1}$ ($g_1(\mathcal{M})$) and a family in $\mathbb{M}_n^{up_2}$ ($g_2(\mathcal{M})$).

Definition 2. *The functions* g_1 *and* g_2 *from* $\mathbb{M}_{n-1}$ *to* $\mathbb{M}_n^{up}$ *are given by:*

- $g_1(\mathcal{M}) = \{M \cup \{n-1\} \mid M \in \mathcal{M}\} \cup \emptyset \setminus \{n-1\}$*;*
- $g_2(\mathcal{M}) = \{M \cup \{n-1\} \mid M \in \mathcal{M}\} \cup \emptyset$*.*

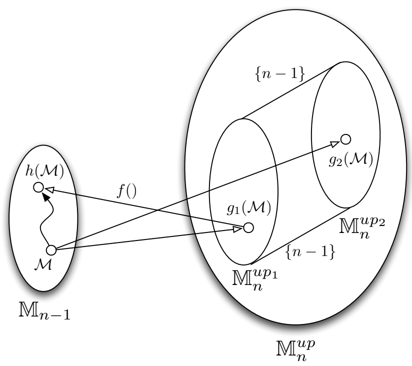


On the left, a Moore co-family of $\mathbb{M}_3$. At the bottom, its image in $\mathbb{M}_4^{up_1}$ by g_1. One can check that all the sets in the family contain the object 3. The singleton $\{3\}$ doesn't belong to the family. At the top on the right, the image by g_2 belongs to $\mathbb{M}_4^{up_2}$ and contains the singleton $\{3\}$.

According to Property 1 below, the maximal family compatible with a family in $\mathbb{M}_n^{up_2}$ spelled $g_2(\mathcal{M})$ is none other than $\mathcal{M}$.

Property 1. Let $\mathcal{M} \in \mathbb{M}_{n-1}$. Then $f(g_2(\mathcal{M})) = \mathcal{M}$.

For the convenience of the study of the maximal family associated with families in $\mathbb{M}_n^{up_1}$, we will denote $h : \mathbb{M}_{n-1} \to \mathbb{M}_{n-1}$ the function $f \circ g_1$. Thus, for all $\mathcal{M}$ in $\mathbb{M}_{n-1}$, $h(\mathcal{M}) = f(g_1(\mathcal{M}))$.



On the left the lattice $\mathbb{M}_{n-1}$ of the Moore co-families on U_{n-1}. At the center, an isomorphic lattice obtained by g_1, on the right another copy obtained by g_2. Their union gives $\mathbb{M}_n^{up}$.

By previous results and the definitions of the functions g_1, g_2, f and h, the set of Moore co-families on U_n can be characterized by the Moore co-families on U_{n-1}. $\downarrow h(\mathcal{M})$ (resp. $\downarrow \mathcal{M}$) is the principal ideal based on $h(\mathcal{M})$ (resp. on $\mathcal{M}$) in the lattice of Moore co-families on U_{n-1}.

Theorem 1. *Let $\mathbb{M}_n$ and $\mathbb{M}_{n-1}$ be the sets of Moore co-families on U_n and U_{n-1} respectively. Then:*

$$\mathbb{M}_n = \bigcup_{\mathcal{M}\in\mathbb{M}_{n-1}} \{g_1(\mathcal{M}) \cup \mathcal{M}', \mathcal{M}' \in\downarrow h(\mathcal{M})\} \cup \bigcup_{\mathcal{M}\in\mathbb{M}_{n-1}} \{g_2(\mathcal{M}) \cup \mathcal{M}'', \mathcal{M}'' \in\downarrow \mathcal{M}\}$$

Note: The case $\mathcal{M} = \{\emptyset\}$ could be extracted from the first term of the decomposition to make the $\mathbb{M}_{n-1}$ set appear. In fact, if $\mathcal{M}$ is reduced to the empty set, $g_1(\mathcal{M})$ contains only the empty set, and the associated maximal family is $2^{U_{n-1}}$. And sets of the sub Moore co-families of $2^{U_{n-1}}$ is exactly $\mathbb{M}_{n-1}$.

Proof : We prove both inclusions:

- $\subseteq$: Let $\mathcal{F} \in \mathbb{M}_n$, $\mathcal{F} = \mathcal{M}_{up} \cup \mathcal{M}_{low}$. Three cases occur :
  1. $\mathcal{M}_{up} = \{\emptyset\}$ then $\mathcal{F}$ belongs to $\mathbb{M}_{n-1}$;
  2. $\mathcal{M}_{up}$ doesn't contain $\{n-1\}$ then let $\mathcal{M} \in \mathbb{M}_{n-1}$ such that $g_1(\mathcal{M}) = \mathcal{M}_{up}$. From Proposition 1 all families compatible with $\mathcal{M}_{up}$ have to be included in $f(g_1(\mathcal{M}))$. Thus the compatible family $\mathcal{M}_{low}$ belongs to $\downarrow h(\mathcal{M})$ and $\mathcal{F}$ belongs to $\{g_1(\mathcal{M}) \cup \mathcal{M}', \mathcal{M}' \in\downarrow h(\mathcal{M})\}$;
  3. $\mathcal{M}_{up}$ contains $\{n-1\}$. This case is similar to the previous one but from Property 1 we know that $f(g_2(\mathcal{M}))$ is equal to $\mathcal{M}$. Then $\mathcal{F}$ belongs to $\{g_2(\mathcal{M}) \cup \mathcal{M}'', \mathcal{M}'' \in\downarrow \mathcal{M}\}$;
- $\supseteq$: From the principle of compatible families and Proposition 1, each element in the left part of the equation is a union-closed family and contains the empty set.

■

3 Study of the h operator

3.1 Some properties of h

Property 2. Let $\mathcal{M} \in \mathbb{M}_n^{up}$ and $M \in \mathcal{M}$. Then $M \setminus \{n-1\} \in f(\mathcal{M})$.

Corollary 1. *For any $\mathcal{M} \in \mathbb{M}_{n-1}$, $\mathcal{M} \subseteq h(\mathcal{M})$.*

Thus the function h behaves as an augmentation operator, idempotent but not monotone. We have actually $\mathcal{M} \subseteq h(\mathcal{M})$ and $h(\mathcal{M}) = h(h(\mathcal{M}))$, but we do not have, for $\mathcal{M} \subseteq \mathcal{M}'$, $h(\mathcal{M}) \subseteq h(\mathcal{M}')$. As a counterexample for $n = 2$, $h(\{\emptyset\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ and $h(\{\emptyset, \{0\}\}) = \{\emptyset, \{0\}\}$. The following property specifies a powerful constraint on the objects that can be added to $\mathcal{M}$ to form $h(\mathcal{M})$.

Property 3. Let $\mathcal{M} \in \mathbb{M}_{n-1}$, $\forall M' \in h(\mathcal{M}) \setminus \mathcal{M}$, $\nexists\ M \in \mathcal{M} \setminus \emptyset$ such that $M \subseteq M'$.

In other words, according to property 3, we can affirm that any set M in $h(\mathcal{M}) \setminus \mathcal{M}$ is a 'quasi-closed' set of $\mathcal{M}$ [2]. Moreover M is incomparable with any set in $\mathcal{M}$ or minimum in $\mathcal{M}$. As a consequence if $\mathcal{M}$ does not have any 'quasi-closed' set, $\mathcal{M}$ is clearly a fixed point for h.

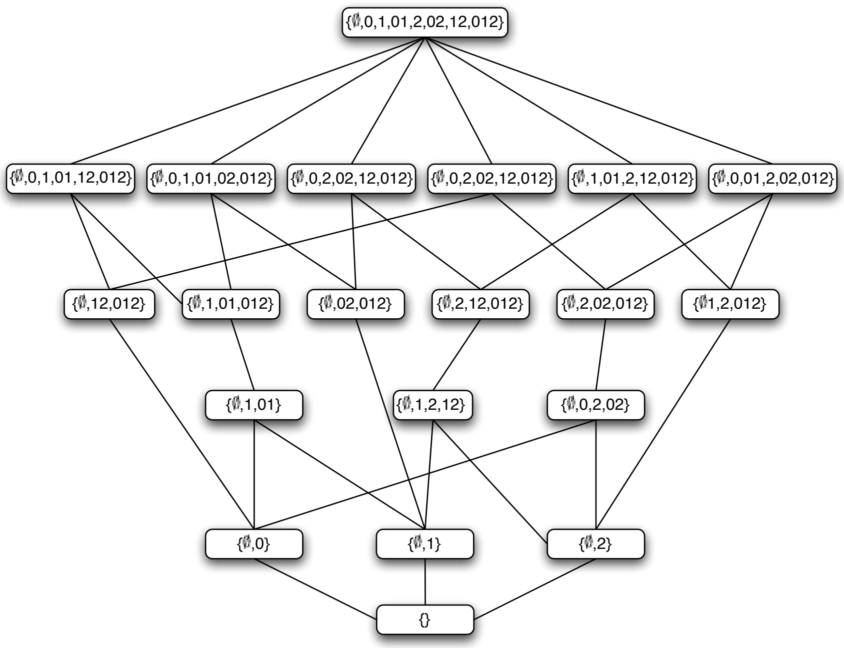


Fig. 4. The set of all fixed points of h in $\mathbb{M}_3$

3.2 Fixed points and equivalence classes induced by the *h* function

The set $\mathbb{M}_{n-1}$ is endowed with the quotient partition associated with h: each class of this partition contains all the families which have the same

[2] The term quasi-closed set has to be used in a context of closed system. So, in our context, we mean that M is a 'quasi-closed' set of $\mathcal{M}$ if the union of M with any set in $\mathcal{M}$ already belongs to $\mathcal{M}$.

image by h. By Proposition 1 (unicity of maximal family) and due to the fact that h is an augmentation operator ($\mathcal{M} \subseteq h(\mathcal{M})$), each fixed point associated to h is a unique representative of each class and corresponds to its upper bound.

Definition 3. *Let $\mathcal{C}(\mathcal{M})$ be an equivalence class of $\mathcal{M}$ in $\mathbb{M}_{n-1}$ such that:* $\mathcal{C}(\mathcal{M}) = \{\mathcal{M}' \in \mathbb{M}_{n-1} \mid h(\mathcal{M}) = h(\mathcal{M}')\}$.

In the following Corollary 2 we show that each class induced by h has a distributive lattice structure. In other terms, a class induced by h is closed under both union and intersection.

Property 4. Let $\mathcal{M}$, $\mathcal{M}'$ in $\mathbb{M}_{n-1}$ such that $h(\mathcal{M}) = h(\mathcal{M}')$. Then:

1. $\mathcal{M} \cup \mathcal{M}'$ is a Moore co-family.
2. $h(\mathcal{M} \cap \mathcal{M}') = h(\mathcal{M}) = h(\mathcal{M}')$.
3. $h(\mathcal{M} \cup \mathcal{M}') = h(\mathcal{M}) = h(\mathcal{M}')$.

Corollary 2. *For any $\mathcal{M}$ in $\mathbb{M}_{n-1}$, $(\mathcal{C}(\mathcal{M}), \subseteq)$ is a distributive lattice.*

4 Conclusion

In this paper, we stated a decomposition theorem of the Moore co-families lattice. We studied the h operator relied upon by the theorem. Particularly, we showed that families assuming the same value for h have a distributive lattice structure. Then, it would be interesting to characterize irreducible elements of these lattices. We are currently investigating why two families are in the same class using different representations of closure systems (irreducible elements, implicational basis, ...)

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5 Appendix

Proof of Property 1 : by double inclusion :

- $\mathcal{M} \subseteq f(g_2(\mathcal{M}))$. Let $M \in \mathcal{M}$, by definition of g_2, we have $M \cup \{n-1\} \in g_2(\mathcal{M})$. From Property 2 applied to $M \cup \{n-1\}$ in $g_2(\mathcal{M})$, we have $(M \cup \{n-1\}) \backslash \{n-1\} \in f(g_2(\mathcal{M}))$. In other words, $M \in f(g_2(\mathcal{M}))$.
- $f(g_2(\mathcal{M})) \subseteq \mathcal{M}$. Suppose that $M \in f(g_2(\mathcal{M}))$ such that $M \notin \mathcal{M}$. By construction of f, we have $\forall M' \in g_2(\mathcal{M})$, $M \cup M' \in g_2(\mathcal{M})$. Now, by the definition of g_2, $M' = \{n-1\} \in g_2(\mathcal{M})$ and so $M \cup \{n-1\} \in g_2(\mathcal{M})$. By the construction of g_2, we obtain $M \in \mathcal{M}$. Contradiction.

Proof of Property 2: Suppose that $M \in \mathcal{M}$ and $M \backslash \{n-1\} \notin f(\mathcal{M})$. From the definition of f: $\exists\, M' \in \mathcal{M}$ such that $M \backslash \{n-1\} \cup M' \notin \mathcal{M}$. Now $\forall\, M' \in \mathcal{M}$, $M' \cup (M \backslash \{n-1\}) = M' \cup M$ ($\{n-1\} \in M'$ since $\mathcal{M} \in \mathbb{M}_n^{up}$). Moreover, $M' \cup M \in \mathcal{M}$ since $\mathcal{M}$ is a Moore co-family containing M and M'. Contradiction.
Proof of Corollary 1: Let us show that $\forall M \in \mathcal{M}$, $\mathcal{M} \in h(\mathcal{M})$. From the definition of $g_1 : \forall M \in \mathcal{M}$, $M \cup \{n-1\} \in g_1(\mathcal{M})$. From Property 2, we have for $M \cup \{n-1\} \in g_1(\mathcal{M})$, $(M \cup \{n-1\}) \backslash \{n-1\} \in f(g_1(\mathcal{M})) = h(\mathcal{M})$.
Proof of Property 3: Suppose there exists $M \in \mathcal{M} \backslash \emptyset$ such that $M \subseteq M'$ with $M' \in h(\mathcal{M}) \setminus \mathcal{M}$. By construction of g_1, we have $M \cup \{n-1\} \in g_1(\mathcal{M})$. Moreover, since $M' \in h(\mathcal{M}) \setminus \mathcal{M}$, we deduce that $\forall X \in g_1(\mathcal{M})$, $X \cup M' \in g_1(\mathcal{M})$. So, since $M \cup \{n-1\} \in g_1(\mathcal{M})$ we have $M \cup \{n-1\} \cup M' \in g_1(\mathcal{M})$. Moreover, since $M \subset M'$, we have $M \cup \{n-1\} \cup M' = M' \cup \{n-1\} \in g_1(\mathcal{M})$. From the construction of g_1, we obtain $M' \in \mathcal{M}$ which is a contradiction with the hypothesis. Contradiction.
Proof of Property 4:

1. **Lemma 1.** *Let* $\mathcal{M}$*,* $\mathcal{M}' \in \mathbb{M}_{n-1}$ *such that* $h(\mathcal{M}) = h(\mathcal{M}')$*. Then* $\forall M \in \mathcal{M}$ *et* $\forall M' \in \mathcal{M}'$ *we have* $M \cup M' \in \mathcal{M} \cap \mathcal{M}'$*.*
   Proof of lemma: Since M' belongs to $h(\mathcal{M}') = h(\mathcal{M})$, M' belongs to $h(\mathcal{M})$. From the definition, $h(\mathcal{M})$ is the set of objects such that their own union with M is in $\mathcal{M}$. In particular we do have $M \cup M'$ in $\mathcal{M}$. We do have too $M \cup M'$ in $\mathcal{M}'$. So $M \cup M'$ belongs to $\mathcal{M} \cap \mathcal{M}'$. ■.
   Then, let $\mathcal{M}$, $\mathcal{M}'$ in $\mathbb{M}_{n-1}$ such that $h(\mathcal{M}) = h(\mathcal{M}')$, let M and M' in $\mathcal{M} \cup \mathcal{M}'$:
   - if M and M' are in $\mathcal{M}$, $M \cup M'$ belongs to $\mathcal{M} \cup \mathcal{M}'$ since $\mathcal{M}$ is a Moore co-family;
   - if M and M' are in $\mathcal{M}'$, $M \cup M'$ belongs to $\mathcal{M} \cup \mathcal{M}'$ since $\mathcal{M}'$ is a Moore co-family;
   - if M belongs to $\mathcal{M}$ and M' to $\mathcal{M}'$ then from the lemma $M \cup M'$ belongs to $\mathcal{M} \cap \mathcal{M}' \subseteq \mathcal{M} \cup \mathcal{M}'$;

   So for any couple M and M' in $\mathcal{M} \cup \mathcal{M}'$, $M \cup M'$ belongs to $\mathcal{M} \cup \mathcal{M}'$. $\mathcal{M} \cup \mathcal{M}'$ is a Moore co-family.
2. by double inclusion:
   - Let us show that $h(\mathcal{M} \cap \mathcal{M}') \subseteq h(\mathcal{M})$ if $h(\mathcal{M}) = h(\mathcal{M}')$.
     Let us suppose there exists X in $h(\mathcal{M} \cap \mathcal{M}'))$ with X which does not belongs to $h(\mathcal{M}) = h(\mathcal{M}')$. Since $X \notin h(\mathcal{M})$, there exists M_1 in $\mathcal{M}$ such that $X \cup M_1$ is not in $\mathcal{M}$. By hypothesis $h(\mathcal{M}) = h(\mathcal{M}')$ which means that $X \cup M_1$ does not belong to $h(\mathcal{M}')$. Then, from the definition of h again, there exists M_1' in $\mathcal{M}'$ such that $X \cup M_1 \cup M_1'$ is not in $\mathcal{M}'$.

     So, we can construct two incompatible schemas (cf. schema in figure 5):
     - a) We just showed the existence of M_1 in $\mathcal{M}$ and of M_1' in $\mathcal{M}'$ such that $X \cup M_1 \cup M_1'$ does not belongs to $\mathcal{M} \cup \mathcal{M}'$. Moreover, from Property 3, $X \cup M_1 \cup M_1'$ can't belong to $h(\mathcal{M}) = h(\mathcal{M}')$.
     - b) From the lemma we do have $M_1 \cup M_1'$ in $\mathcal{M} \cap \mathcal{M}'$, since X belongs to $(h(\mathcal{M} \cap \mathcal{M}'))$, $X \cup M_1 \cup M_1'$ belongs to $\mathcal{M} \cap \mathcal{M}'$ (from definition of h);

     The existence hypothesis of X in $h(f(\mathcal{M} \cap \mathcal{M}'))$ with X not in $h(\mathcal{M}) = h(\mathcal{M}')$ leads to two incompatible schemas. Contradiction.
   - Let us show that $h(\mathcal{M}) \subseteq h(\mathcal{M} \cap \mathcal{M}')$ if $h(\mathcal{M}) = h(\mathcal{M}')$.
     Let $X \in h(\mathcal{M})$, from the definition of h, we have $\forall M \in \mathcal{M}$, $M \cup X \in \mathcal{M}$. The same: since $h(\mathcal{M}) = h(\mathcal{M}')$, we do have $\forall M' \in \mathcal{M}'$, $M' \cup X \in \mathcal{M}'$. As a consequence, $\forall M \in \mathcal{M} \cap \mathcal{M}'$, $M \cup X \in \mathcal{M} \cap \mathcal{M}'$. From the definition of h again, we must have $X \in h(\mathcal{M} \cap \mathcal{M}')$.

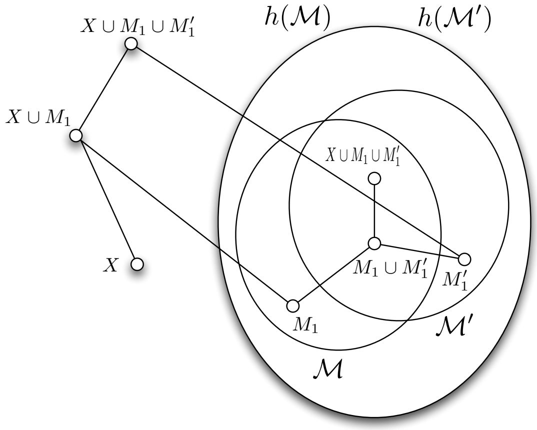

Fig. 5. On the left, from schema a) we construct the object $X \cup M_1 \cup M'_1$ outside of $h(\mathcal{M}) = h(\mathcal{M}')$. In the middle, from schema b) we construct the same object in $\mathcal{M} \cap \mathcal{M}'$. Contradiction.

3. By double inclusion :
   - Let us show that $h(\mathcal{M} \cup \mathcal{M}') \subseteq h(\mathcal{M})$ if $h(\mathcal{M}) = h(\mathcal{M}')$.
     Let us suppose there exists X in $h(\mathcal{M} \cup \mathcal{M}')$ with X not in $h(\mathcal{M}) = h(\mathcal{M}')$. Since $X \notin h(\mathcal{M})$, there exists M_1 in $\mathcal{M}$ such that $X \cup M_1$ is not in $\mathcal{M}$.
     So, we can design two incompatible schemas of construction (cf. schema in figure 6):
     - a) We just showed the existence of M_1 in $\mathcal{M}$ such that $X \cup M_1$ is not in $\mathcal{M}$. Moreover, from Property 3, $X \cup M_1$ can't belong to $h(\mathcal{M}) = h(\mathcal{M}')$.
     - b) Since X belongs to $h(\mathcal{M} \cup \mathcal{M}')$, for any object of $\mathcal{M}$ the union of X with this object belongs to $\mathcal{M} \cup \mathcal{M}'$. In particular for M_1 we do have $X \cup M_1$ in $\mathcal{M} \cup \mathcal{M}'$.

     The existence hypothesis of X in $h(f(\mathcal{M} \cup \mathcal{M}'))$ with X not in $h(\mathcal{M}) = h(\mathcal{M}')$ leads to two incompatible schemas. Contradiction.
   - Let us show that $h(\mathcal{M}) \subseteq f(\mathcal{M} \cup \mathcal{M}')$ if $h(\mathcal{M}) = h(\mathcal{M}')$.
     Let $X \in h(\mathcal{M})$, from the definition of h, we have $\forall M \in \mathcal{M}$, $M \cup X \in \mathcal{M}$. In the same way, since $h(\mathcal{M}) = h(\mathcal{M}')$, we have $\forall M' \in \mathcal{M}'$, $M' \cup X \in \mathcal{M}'$. As a consequence, $\forall M \in \mathcal{M} \cup \mathcal{M}'$, $M \cup X \in \mathcal{M} \cup \mathcal{M}'$. From the definition of h again, we do have $X \in h(\mathcal{M} \cup \mathcal{M}')$.

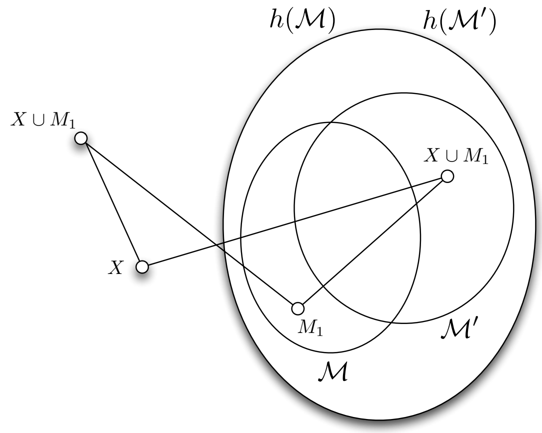

Fig. 6. On left, from schema a) we construct the object $X \cup M_1$ outside of $h(\mathcal{M}) = h(\mathcal{M}')$. In the middle, from schema b) we construct the same object in $\mathcal{M} \cup \mathcal{M}'$. Contradiction.